# The vorticity jump across a gasdynamic discontinuity

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## SUMMARY

An expression is obtained for the jump in the vorticity across a gasdynamic discontinuity in an inviscid flow. This result generalizes results of Truesdell (1952) and Lighthill (1957) for the vorticity behind a steady curved shock in a uniform flow and that of Emmons (1957) for the vorticity jump across a steady flame. The derivation is a dynamical one, and no assumptions on the composition or thermodynamic properties of the fluid are made. The jump in vorticity in the steady flow case is found to depend upon the jump in density and upon gradients along the discontinuity surface of the tangential velocity component and of the normal mass flow. An analogous result is found with unsteady flow.

## INTRODUCTION

It is a well-known fact that the vorticity in a steady compressible flow is generally discontinuous across a shock wave. Truesdell (1952) first obtained a general expression for the vorticity behind a curved two-dimensional steady shock in a uniform flow. Truesdell states his result thus: *the magnitude of the vorticity generated by a shock of given strength and curvature depends only on the magnitude of the tangential component of velocity and is independent of the form of the equation of state.* Here the strength of a shock is defined as the density jump across the shock divided by the density in front. Other investigators, including this writer and Lighthill, unaware of Truesdell's work, later rederived the result. However, Lighthill (1957, pp. 14, 15) also provided a significant generalization in showing that the result was valid, when expressed in terms of the axes of principal curvature, for a steady shock wave of general shape in a uniform flow. Lighthill specified the density ratio across the shock to be the limiting value for a very strong shock, but this restriction is not used in his analysis and is unnecessary.

The vorticity relations just discussed depend upon a number of assumptions, of which the most restrictive is that the flow in front of the shock is uniform. The methods used all require the application of Crocco's vorticity law relating the vorticity with the entropy gradient, and the assumptions which underlie this law must be made. These assumptions are that the flow is steady, is isocompositional, and is isoenergetic (i.e. has constant total enthalpy).

A similar question arises in the theory of the propagation of a laminar flame in a combustible gas mixture, as to the magnitude of the vorticity engendered by the flame. In this case the disturbance caused by the flame is felt in the fluid region in front of the flame, and the assumption of uniform flow in this region cannot be made. Emmons (1957) has obtained an expression for the change in vorticity across a flame in terms of the density ratio across the discontinuity and the jump in tangential entropy gradient. Emmons uses a modified form of Crocco's vorticity law in his derivation, and his results are also restricted to steady and isocompositional flows. Emmons's results and the results of this paper were obtained independently and are not directly comparable.

The simplicity of Truesdell's relation and its independence of the thermodynamic state of the fluid suggests that the relation is a purely dynamical one. A purely dynamical relation must be derivable without recourse to a thermodynamic law such as Crocco's vorticity law. In presenting a dynamical derivation for the vorticity jump we shall consider first the case of steady flow, in which the principal features of the derivation are not obscured by the complexity of an arbitrarily moving discontinuity. Our method involves a vector decomposition normal to and tangential to the discontinuity surface. The discontinuities considered are gasdynamic ones, including shock waves, deflagrations, and detonations, but excluding contact discontinuities or slip streams.

### THE VORTICITY JUMP IN STEADY FLOW

The discontinuity surface is assumed to be sufficiently smooth and continuous, without discontinuities in slope. The orientation of the surface is specified at any point by the unit normal vector $\mathbf{n}$, which is assumed to be differentiable along the surface. A natural local coordinate system is imagined, based on any intrinsic two-dimensional coordinate system on the surface. The additional coordinate $n$ is distance along a straight normal from the discontinuity surface, with the corresponding coordinate surfaces geodesic parallels to the discontinuity surface.

The unit vector $\mathbf{n}$ is the gradient of this normal coordinate, and we obtain immediately

$$\nabla \times \mathbf{n} = 0. \tag{1}$$

The negative of the gradient of $\mathbf{n}$ is the two-dimensional curvature tensor in the discontinuity surface

$$\mathfrak{K} = -\nabla \mathbf{n}, \tag{2}$$

which from (1) must be symmetric. Because $\mathbf{n}$ is a unit vector $\nabla\mathbf{n}$ can have no component normal to the surface. The curvature is defined so as to be positive if the surface is concave on the side from which $\mathbf{n}$ is directed.

In general, all vectors will be considered in a form decomposed, with respect to the surface, into a normal component and a tangential two-dimensional vector. Thus, with the velocity vector $\mathbf{q}$ for the flow field, we have

$$\mathbf{q} = \mathbf{n}q_n + \mathbf{q}_t, \tag{3}$$

where

$$q_n = \mathbf{n}\,.\,\mathbf{q}, \qquad \mathbf{q}_t = \mathbf{n}\times(\mathbf{q}\times\mathbf{n}). \tag{4}$$

The vorticity is the curl of $\mathbf{q}$, and is expressed as

$$\begin{aligned}\boldsymbol{\zeta} &= \nabla\times(\mathbf{n}q_n) + \nabla\times\mathbf{q}_t \\ &= \mathbf{n}(\nabla\times\mathbf{q}_t)_n + (\nabla\times\mathbf{q}_t)_t - \mathbf{n}\times\nabla q_n\,;\end{aligned} \tag{5}$$

we note that

$$\zeta_n = (\nabla\times\mathbf{q}_t)_n. \tag{6}$$

Vector expansion of the identity $\nabla(\mathbf{n}\,.\,\mathbf{q}_t) = 0$ gives us the result

$$(\nabla\times q_t)\times\mathbf{n} = \mathbf{n}\,.\,\nabla\mathbf{q}_t + \mathbf{q}_t\,.\,\nabla\mathbf{n}, \tag{7}$$

and from (5) we obtain an expression for $\boldsymbol{\zeta}_t$

$$\boldsymbol{\zeta}_t = \mathbf{n}\times\left(\frac{\partial\mathbf{q}_t}{\partial n} - \mathbf{q}_t\,.\,\mathbb{K} - \nabla_t q_n\right), \tag{8}$$

in which the subscript $t$ on the nabla operator $\nabla$ indicates that only the tangential part of the derivative is included.

We need also an expression for the tangential part of the directional derivative of the velocity, $\mathbf{q}\,.\,\nabla\mathbf{q}$. A straightforward analysis starting with (3) gives

$$(\mathbf{q}\,.\,\nabla\mathbf{q})_t = q_n\left(\frac{\partial\mathbf{q}_t}{\partial n} - \mathbf{q}_t\,.\,\mathbb{K}\right) + \mathbf{q}_t\,.\,\nabla_t\mathbf{q}_t. \tag{9}$$

We now turn to the steady discontinuity relations, with the symbol $\delta$ indicating the jump of a quantity across the discontinuity. The continuity condition is

$$\delta(\rho q_n) = 0, \tag{10}$$

while the tangential momentum condition is

$$\delta\mathbf{q}_t = 0. \tag{11}$$

It is at this point, in establishing (11), that we assume non-zero mass flow through the discontinuity and exclude contact discontinuities from our consideration. The normal momentum condition is

$$-\delta p = \rho q_n\,\delta q_n. \tag{12}$$

The energy condition across the discontinuity is not used. Since relations (10), (11) and (12) hold everywhere on the discontinuity surface we may differentiate them along the surface. Also, the operators $\delta$ and $\nabla_t$ are commutable. The increment in the vorticity is

$$\delta\zeta_n = 0, \tag{13}$$

$$\delta\boldsymbol{\zeta}_t = \mathbf{n}\times\delta\left(\frac{\partial\mathbf{q}_t}{\partial n} - \nabla_t q_n\right), \tag{14}$$

from (6) and (8). Equation (13) is a statement of the obvious fact that the normal component of the vorticity is continuous across a discontinuity.

We next write the tangential part of the momentum equation for the fluid flow. Since the flow is assumed steady, we obtain immediately with the aid of (9)

$$-\nabla_t p = \rho q_n\left(\frac{\partial \mathbf{q}_t}{\partial n} - \mathbf{q}_t \cdot \mathbf{K}\right) + \rho \mathbf{q}_t \cdot \nabla_t \mathbf{q}_t. \tag{15}$$

Our method now is to equate the tangential gradient of (12) with the jump in (15);

$$\begin{aligned} -\delta\nabla_t p &= (\rho q_n)\delta\nabla_t q_n + \nabla_t(\rho q_n)\delta q_n \\ &= \rho q_n\,\delta\left(\frac{\partial \mathbf{q}_t}{\partial n}\right) + \mathbf{q}_t \cdot \nabla_t\, \mathbf{q}_t\,\delta\rho. \end{aligned} \tag{16}$$

Combining (10), (14) and (16) gives us our desired expression for $\delta\boldsymbol{\zeta}_t$;

$$\delta\boldsymbol{\zeta}_t = \mathbf{n} \times [\nabla_t(\rho q_n)\delta(\rho^{-1}) - (\rho q_n)^{-1}\mathbf{q}_t \cdot \nabla_t \mathbf{q}_t\,\delta(\rho)]. \tag{17}$$

In this expression the only jumps which appear are jumps in the density, and these are multiplied by quantities which are continuous across the discontinuity. Because of (13), the subscript $t$ on $\boldsymbol{\zeta}$ in (17) is not needed.

Finally, we obtain Truesdell's expression with Lighthill's generalization, taking the flow in front of the discontinuity to be uniform with density $\rho_0$ and velocity $\mathbf{U}$. We evaluate the quantities appearing in (17),

$$\rho q_n = \rho_0 U_n, \tag{18a}$$

$$\nabla_t(\rho q_n) = \rho_0 \mathbf{U} \cdot \nabla \mathbf{n} = -\rho_0 \mathbf{U}_t \cdot \mathbf{K}, \tag{18b}$$

$$\mathbf{q}_t \cdot \nabla_t \mathbf{q}_t = \mathbf{n} \times (\mathbf{U} \times \mathbf{U}_t \cdot \nabla \mathbf{n}) = U_n \mathbf{U}_t \cdot \mathbf{K}, \tag{18c}$$

and obtain

$$\delta\boldsymbol{\zeta}_t = \mathbf{n} \times \mathbf{U}_t \cdot \mathbf{K}(\rho_1^{-1} - \rho_0^{-1})\delta\rho, \tag{19}$$

or, since the flow in front of the discontinuity is irrotational,

$$\boldsymbol{\zeta}_1 = -\frac{(1-\epsilon)^2}{\epsilon}\mathbf{n} \times \mathbf{U}_t \cdot \mathbf{K}. \tag{20}$$

Here $\rho_1$ is the density behind the discontinuity, and

$$\epsilon = \rho_0/\rho_1 \tag{21}$$

is the density ratio across the discontinuity. It should be noted that this result is not limited to shock waves, but would hold also for a condensation shock or detonation.

## THE VORTICITY JUMP IN UNSTEADY FLOW

In dealing with the unsteady flow case we must take into account not only the unsteady nature of the flow field but also an arbitrary motion of the discontinuity surface. Many of the relations obtained for the steady flow case are still valid, principally the kinematic relations (6), (8) and (9), and the jump relations for the tangential velocity and the vorticity (11), (13) and (14). The primary modification which must be made is to the relations (10) and (12) involving the normal mass flow across the discontinuity.

Of the available ways in which the shock motion may be specified, the most convenient is in terms of its normal velocity $\mathbf{n}q_s$. We use the notation $d/dt$ to denote the time derivative taken at a point which always lies on the discontinuity and which moves in a direction normal to the discontinuity when the discontinuity moves. If $\boldsymbol{\omega}_s$ is the angular velocity of the surface measured at such a point and is defined as a tangential vector, we may show that

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega}_s \times \mathbf{n} = -\nabla_t \, q_s. \tag{22}$$

We designate the relative normal velocity component $q_r$ as

$$q_r = q_n - q_s, \tag{23}$$

and obtain the discontinuity relations

$$\delta(\rho q_r) = 0 \tag{24}$$

in place of (10), and

$$-\delta p = (\rho q_r)\delta q_r = (\rho q_r)\delta q_n \tag{25}$$

in place of (12). We use the symbol $D$ to designate the classical total or material derivative with respect to time; and we introduce the symbol $D_t$ to designate a tangential total time derivative, the time derivative with respect to an observer who moves along the shock with a total velocity $\mathbf{n}q_s + \mathbf{q}_t$. If such a derivative is a vector, we shall include only its tangential part. Note that for such an observer the discontinuity is a normal one. The distinction between the derivatives $D$ and $D_t$ is immaterial when they are applied to the vector $\mathbf{n}$, because the space derivative of $\mathbf{n}$ in the normal direction is zero. Thus we have

$$D\mathbf{n} = D_t \, \mathbf{n} = \frac{d\mathbf{n}}{dt} + \mathbf{q}_t \cdot \nabla \mathbf{n}. \tag{26}$$

For the tangential velocity $\mathbf{q}_t$ we have

$$\frac{d\mathbf{q}_t}{dt} = \frac{\partial \mathbf{q}_t}{\partial t} + q_s \frac{\partial \mathbf{q}_t}{\partial n}, \tag{27}$$

and

$$D_t \mathbf{q}_t = (D\mathbf{q}_t)_t - q_r \frac{\partial \mathbf{q}_t}{\partial n} = \left(\frac{d\mathbf{q}_t}{dt}\right)_t + \mathbf{q}_t \cdot \nabla_t \mathbf{q}_t. \tag{28}$$

In place of (15) we may now write

$$\begin{aligned} -\nabla_t \, p &= \rho(D\mathbf{n}q_n)_t + \rho(D\mathbf{q}_t)_t \\ &= \rho(q_r + q_s)D_t \, \mathbf{n} + \rho q_r \frac{\partial \mathbf{q}_t}{\partial n} + \rho D_t \, \mathbf{q}_t. \end{aligned} \tag{29}$$

Applying the same procedure as was used in the case of steady flow, we obtain the desired expression for the vorticity jump as

$$\delta \boldsymbol{\zeta}_t = \mathbf{n} \times [\nabla_t(\rho q_r)\delta(\rho^{-1}) - (\rho q_r)^{-1}(D_t \, \mathbf{q}_t + q_s \, D_t \mathbf{n})\delta(\rho)] \tag{30}$$

in place of (17). The quantity $D_t\mathbf{n}$ is given in (26), or, with (2) and (22), may be re-expressed as

$$D_t \mathbf{n} = -\nabla_t q_s - \mathbf{q}_t \cdot \mathbb{K}. \tag{31}$$

The quantity $D_t \mathbf{q}_t$ is given by (28) with (27). It may be checked that (30) is invariant under a velocity transformation to another unaccelerated frame of reference. Again, the subscript $t$ on $\boldsymbol{\zeta}$ in (30) is not needed.

As an example of our result (30) for unsteady flow we apply it to the case in which the flow in front of the discontinuity is at a uniform density $\rho_0$ and uniform velocity **U**. We evaluate the quantities appearing in (30) to obtain, in analogy with (18),

$$\rho q_r = \rho_0(U_n - q_s), \tag{32a}$$

$$\nabla_t(\rho q_r) = \rho_0 D_t \mathbf{n}, \tag{32b}$$

$$D_t \mathbf{q}_t = \mathbf{n} \times (\mathbf{U} \times D_t \mathbf{n}) = -U_n D_t \mathbf{n}. \tag{32c}$$

The vorticity jump is then obtained from (30),

$$\delta \boldsymbol{\zeta}_t = \mathbf{n} \times (\mathbf{U}_t . \mathbb{K} + \nabla_t q_s)(\rho_1^{-1} - \rho_0^{-1})\delta\rho, \tag{33}$$

in place of (19), or

$$\zeta_1 = -\frac{(1-\epsilon)^2}{\epsilon}\mathbf{n} \times (\mathbf{U}_t . \mathbb{K} + \nabla_t q_s) \tag{34}$$

in place of (20), with $\epsilon$ given as before by (21). This result is the generalization to unsteady flow of the Truesdell–Lighthill vorticity expression. It may be noted that the fundamental quantity appearing in (20) or (34) is the angular velocity of the discontinuity surface with respect to the observer for whom a derivative is $D_t$. We may rewrite (20) or (34) in the form

$$\zeta_1 = \frac{(1-\epsilon)^2}{\epsilon}\boldsymbol{\omega}, \tag{35}$$

where the angular velocity $\boldsymbol{\omega}$ is

$$\boldsymbol{\omega} = \mathbf{n} \times D_t \mathbf{n} \tag{36a}$$

$$= \boldsymbol{\omega}_s + \mathbf{U}_t . \mathbb{K} \times \mathbf{n}, \tag{36b}$$

with $\boldsymbol{\omega}_s$ defined in (22).

## References

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